Discrete Ziggurat: A Time-Memory Trade-off for Sampling from a Gaussian Distribution over the Integers

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Selected Areas in Cryptography
Aug 16, 2013

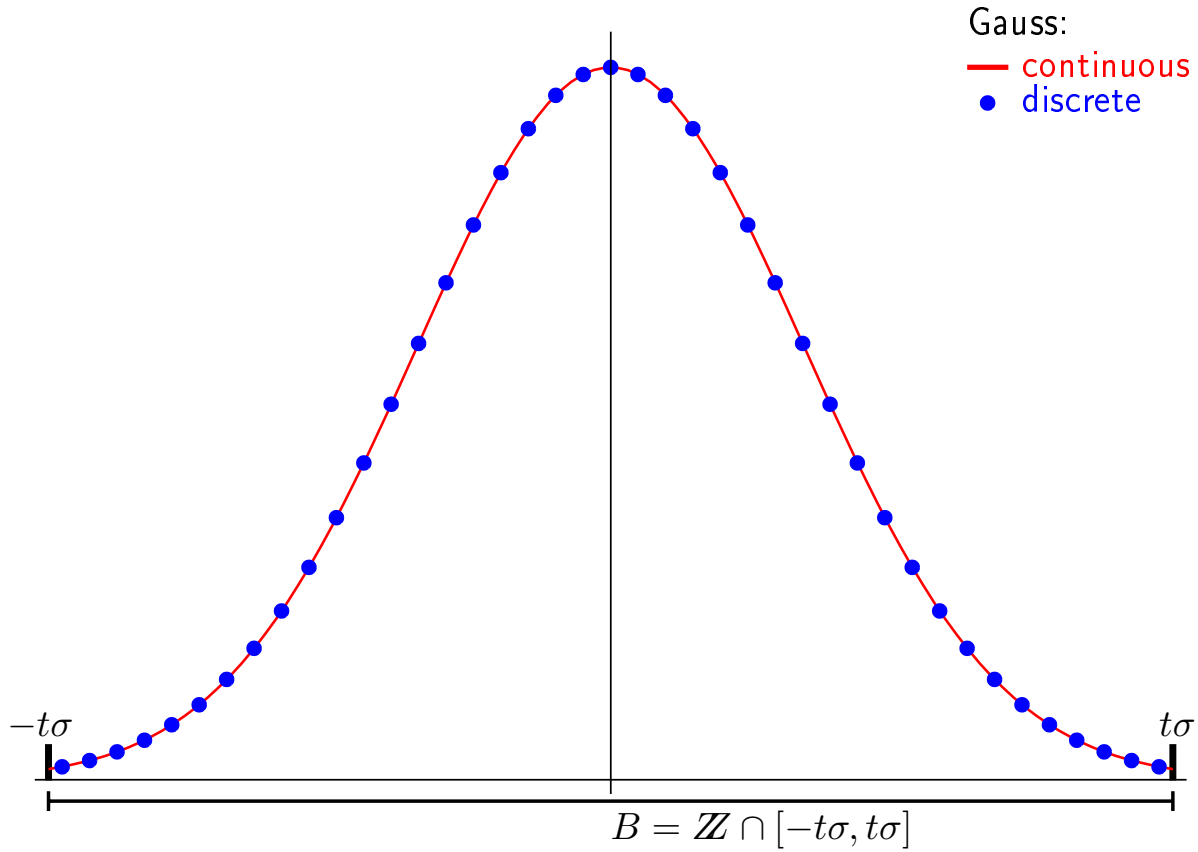
Outline

Motivation and Contribution

- Discrete Gaussians widely used in lattice-based crypto
  - E.g. signatures, encryption, (F)HE, multilinear maps
- Critical technical challenge: accurate and efficient sampling of discrete Gaussians
  - E.g. sampling $\approx 50\%$ of signing time [WHCB13]
- Existing methods: either large memory or very slow
  - E.g. Peikert's sampler about 12MB of storage [GD12]
  - No flexibility in choice of memory and speed
  - Memory requirement acceptable on PC, but not on smaller devices
- Our contribution: alternative sampler for discrete Gaussians offering a flexible trade-off between speed and memory

Discrete Gaussians and Samplers

- Discrete Gaussian distribution D_σ for parameter σ assigns $x \in \mathbb{Z}$ probability proportional to $\rho_\sigma(x) = \exp(-\frac{1}{2}x^2/\sigma^2)$
- Sufficient for cryptographic applications: bounded support $B := \mathbb{Z} \cap [-t\sigma, t\sigma]$ with tailcut $t > 0$ large enough [GPV08]

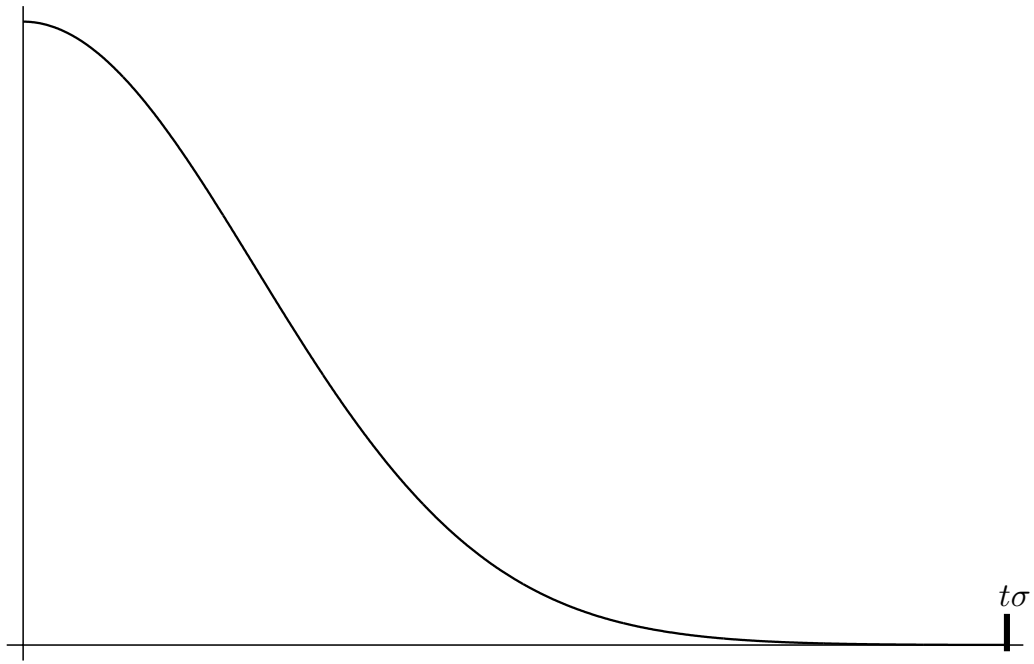


Discrete Gaussians and Samplers

- Rejection sampling (rejSam)
- Inverse cumulative distribution function (invCDF)
- Knuth-Yao (KY)
- Hybrid variants: rejection sampling with lookup-table, ...

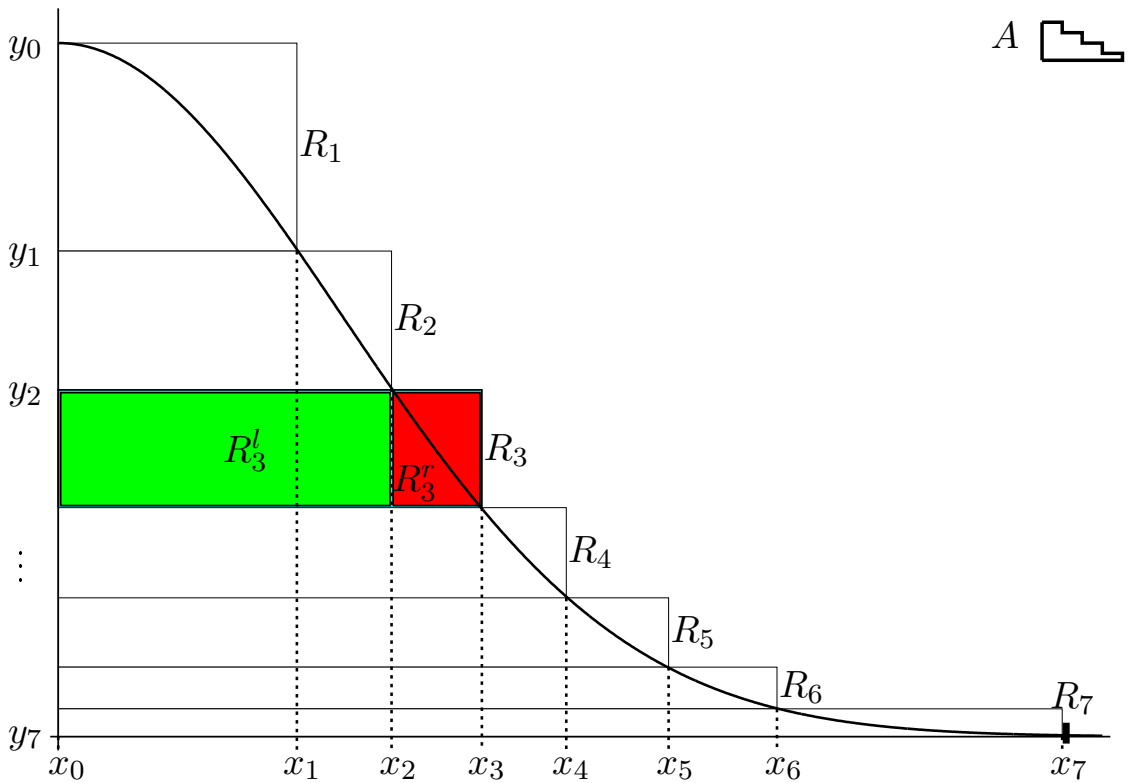
The Ziggurat Algorithm

- Belongs to class of rejection sampling algorithms
- Introduced by Marsaglia and Tsang for sampling from a continuous Gaussian distribution [MT00]
- Observation:
  - Symmetry: sample $x \in [0, t\sigma]$ acc. to PDF
  - Sample sign $s \in \{-1, 1\}$ and return sx
  - Attention: case $x = 0$



The Ziggurat Algorithm

- Sampling $x \in [0, t\sigma]$: Intuition
  - Given: partition of area into rectangles of equal size
  - Choose rectangle $R_i = R_i^l \cup R_i^r$ randomly
  - Sampling in rectangle R_i:
    - Sample $x \in [0, x_i]$ randomly
    - If $x \in R_i^l$: accept x
    - Else sample in R_i^r using rejection sampling (restart)



The Ziggurat Algorithm

- Ziggurat = efficient "instantiation" of rejection sampling in enclosing area A (instead of in $[0, t\sigma] \times [0, 1]$)
- Rectangles of equal size: ensures equality of probabilities
- Storage: (x_i, y_i) for R_i where $i = 1, \ldots, \#\text{rectangles}$
- Expensive part: sampling in R_i^r
- Trade-off:
  - Controlled by #rectangles
  - More rectangles: R_i^l comparatively bigger than R_i^r

  $\rightarrow$ acceptance of x without computing $\rho_\sigma(x)$ with higher probability

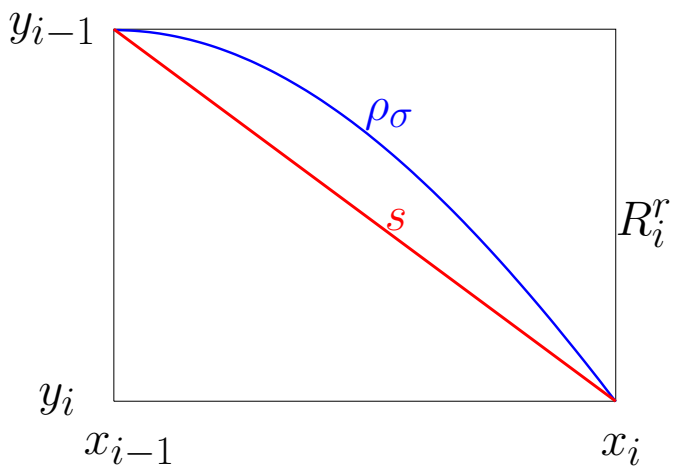
  $\rightarrow$ less rejections of $x \rightarrow$ less 'restarts'
  - But: more memory needed

The Ziggurat Algorithm: Discretization

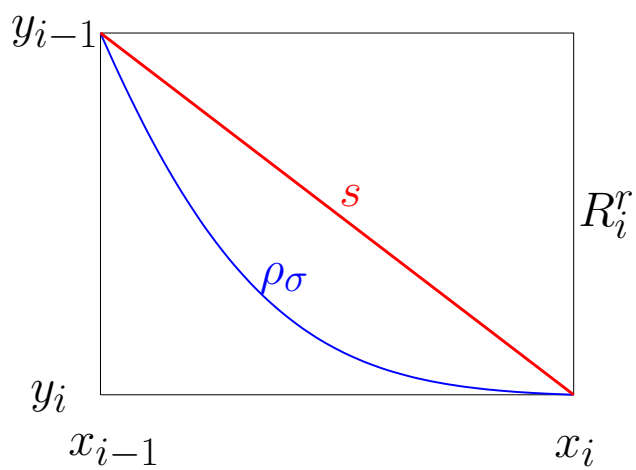
Procedure: same as continuous

Adaptation to discrete case:

- Notion of 'size'
- Pre-computation of rectangles
- Implementation issues:
  - Fix point precision
  - Discretizing the height
- Improvement of sampling in R_i^r: straight line approach



The concave-down case



The concave-up case

Quality of our Sampler and Parameter Choice

Theorem

The statistical distance between the discrete Gaussian distribution D_σ and the distribution $\overline{D}_\sigma$ output by our algorithm is bounded by

$$\Delta(D_\sigma, \overline{D}_\sigma) < te^{(1-t^2)/2} + \frac{|B_0^+|}{\overline{\rho}_\sigma(B^+) + \frac{1}{2}}(2^{-\omega+1} + 2^{-n}).$$

Proof idea: Hybrid argument using intermediary distributions

Quality of our Sampler and Parameter Choice

- Parameters: Gaussian parameter σ, tailcut t, fix point precision n, height precision ω
- Goal: negligible statistical distance, e.g.

$$\underbrace{te^{(1-t^2)/2}}_{l} + \underbrace{\frac{|B_0^+|}{\overline{\rho}_\sigma(B^+) + \frac{1}{2}}(2^{-\omega+1} + 2^{-n})}_{r} < 2^{-100}$$

$\rightarrow$ Find smallest integer t s.t. $l < 2^{-101}$: $t = 13$

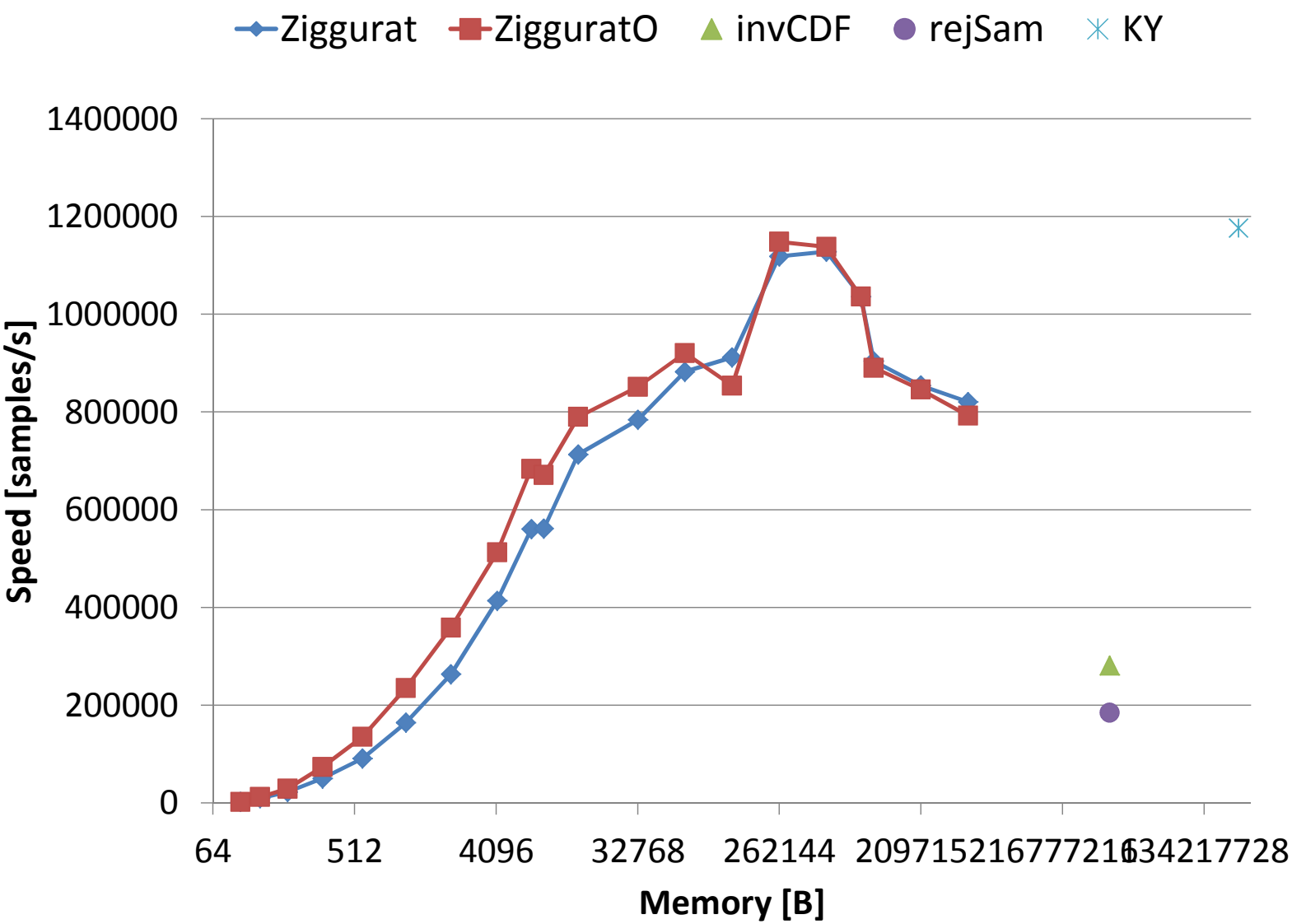
$\rightarrow$ Choose $\omega = n + 1$ reduces complexity of r

$\rightarrow$ Find n such that $r < 2^{-101}$: $n = 106$

Experiments and Results

- C++ implementation using Number Theory Library (NTL, [Sho])
- Parameters: $n = 106$ ($\omega = 107$), $t = 13$, different σ's
- $\sigma = 32$ maintains worst-to-average-case reduction [Reg05], $\sigma = 1.6 \cdot 10^5$ according to [GD12]
- Algorithms: Ziggurat, ZigguratO, invCDF*, rejSam*, KY (* = lookup-table)
- Each algorithm queried to output 10^6 samples
- Measured running time using `clock_gettime` with clock `CLOCK_PROCESS_CPUTIME_ID` (excluded pre-/post-comps.)
- Computed memory consumption using #fixed variables in regard to their type

Experiments and Results



Different samplers for $\sigma = 1.6 \cdot 10^5$

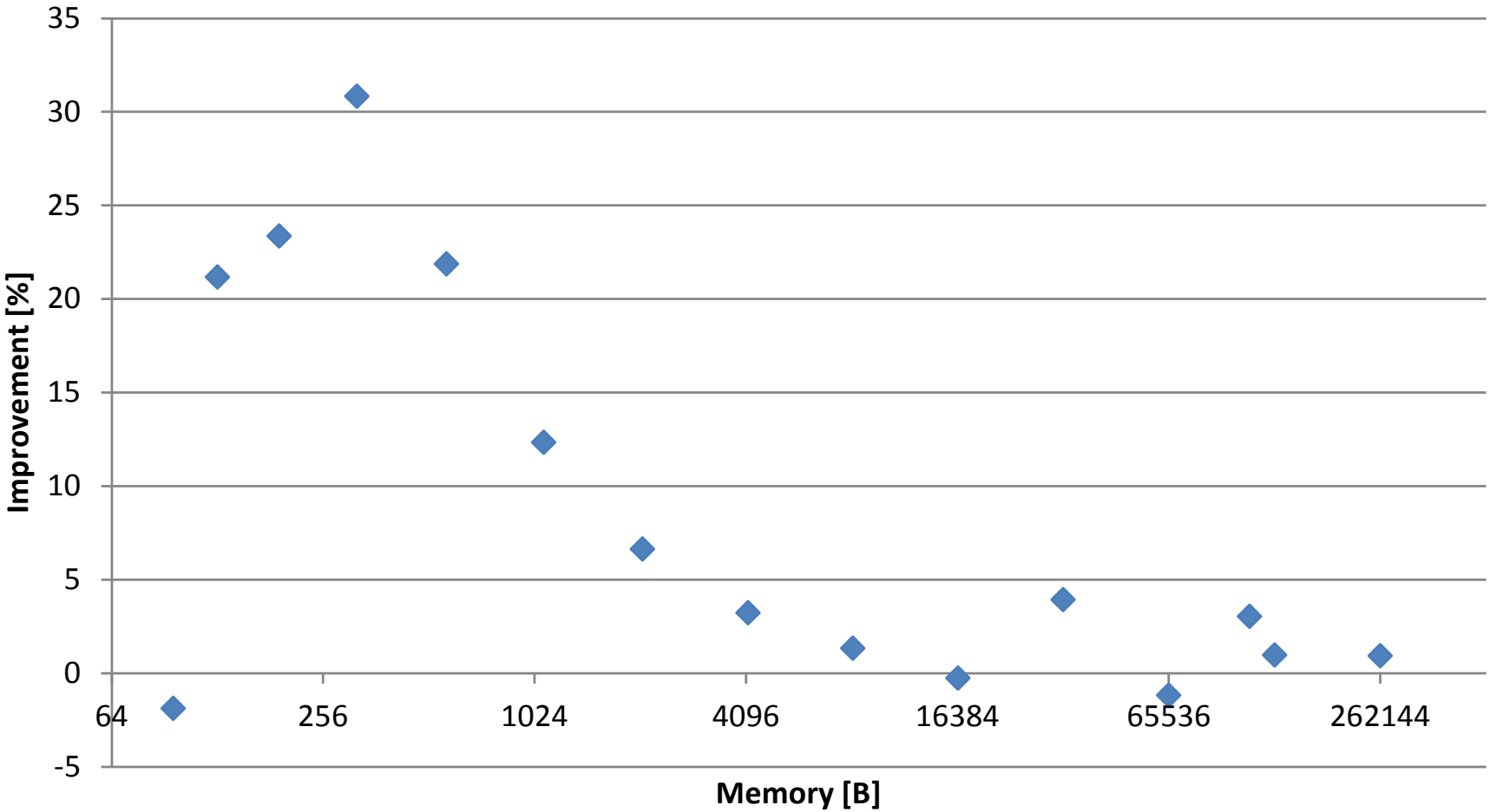
Experiments and Results

Some numbers...

- $\sigma = 32$:
  - rejSam factor 4.2 slower than invCDF, without lookup-table factor 558 slower
  - Ziggurat factor 1.91 slower than invCDF, 2.19 faster than rejSam
  - KY factor 3.53 faster than invCDF, but doubled memory
- $\sigma = 1.6 \cdot 10^5$:
  - invCDF factor 4 slower than Ziggurat, factor 64 more memory
  - rejSam about factor 6 slower than Ziggurat
  - KY only better than Ziggurat by 4%, but 424 times more memory

Experiments and Results

Improvement rate of ZigguratO to Ziggurat



Conclusion: Take-Home-Message

Discrete Ziggurat

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Alternative **sampler** for
discrete Gaussians offering a
flexible **trade-off** between
speed and memory

Further details...

Source code on homepage:
`https://www.cdc.informatik.tu-darmstadt.de/~pschmidt/implementations/ziggurat/ziggurat-src.zip`

Version of paper with proofs on eprint:
`https://eprint.iacr.org/2013/510.pdf`

Thanks!